NEW INVARIANTS FOR A REAL VALUED
AND ANGLE VALUED MAP
(AN ALTERNATIVE TO MORSE-NOVIKOV THEORY)

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This paper summarizes the definitions and the properties of the homotopically relevant invariants associated to a real or an angle valued map in the framework of what we call an Alternative to Morse-Novikov theory. These invariants are configurations of points in the complex plane $\delta_r^f, r \geq 0$, configurations of vector spaces or modules $\hat{\delta}_r^f, r \geq 0$ indexed by complex numbers and collections of Jordan cells $\mathcal{J}_r(f), r \geq 0$. The first are refinements of Betti numbers, the second of homology and the third of monodromy. Although not discussed in this paper, but discussed in works this report is based on, these invariants are computer friendly (*i.e.* can be calculated by computer implementable algorithms when the source of the map is a simplicial complex and the map is simplicial) and are of relevance for the dynamics of flows which admit Lyapunov real or angle valued map.

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1. INTRODUCTION

This paper, except Section 6, is essentially my lecture at The Eighth Congress of Romanian Mathematicians, 2015, Iaşi, Romania.

Classical Morse theory and Morse-Novikov theory consider a Riemannian manifold (M, g) and a Morse real valued or a Morse angle valued map, $f : M \to \mathbb{R}$ or $f : M \to \mathbb{S}^1$, and relate the dynamical invariants of the vector field $grad_g f$, namely

– the rest points of $grad_g f$ = critical points of f,

– the instantons [1] between two rest points x, y of $grad_g f$,

– the closed trajectories of $grad_g f$ (when f is angle valued)

to the algebraic topology of the underlying manifold M or of the pair (M, ξ_f) in case f is angle valued map. Here ξ_f denotes the degree one integral cohomology class represented by f.

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[1] isolated trajectories between critical points

The results of the theory can be applied to any vector field V on M which admits a closed differential one form $\omega \in \Omega^1(M)$ as Lyapunov rather than $grad_g f$, since the dynamics of such vector field V (when generic) is the same as of $grad_g f$ for some Riemannian metric g and some f, angle valued map cf. [3]. The results of the theory can be used in both ways; knowledge of the dynamical invariants of $grad_g f$ permits to calculate the topological invariants of M or of (M, ξ_f) and the algebraic topological invariants of M or of (M, ξ) provide significant constraints for dynamics of a vector field with Lyapunov map representing ξ, cf. [3].

The ANM theory associates to a pair (X, f), X a compact ANR, f a continuous real or angle valued map defined on X and κ a field, a collection of invariants: the configurations δ_r^f, $\hat{\delta}_r^f$, $\hat{\hat{\delta}}_r^f$ and the Jordan cells $\mathcal{J}_r(f), r \geq 0$.

The configuration δ_r^f is a finite collection of points with multiplicity located in $\mathbb{C}$ in case f is real valued and in $\mathbb{C} \setminus 0$ in case f is angle valued. The configuration $\hat{\delta}_r^f$ is given by the same points but instead of natural numbers as multiplicities have κ-vector spaces or free $\kappa[t^{-1}, t]$-modules assigned to, where $\kappa[t^{-1}, t]$ denotes the ring of Laurent polynomials with coefficients in κ. A Jordan cell is a pair (λ, k) with λ a nonzero element in the algebraic closure of the field κ and k a positive integer. The pair (λ, k) is an abbreviation for the $k \times k$ Jordan matrix

$$T(\lambda, k) = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ \cdots & & & & & \\ 0 & 0 & \cdots & 0 & \lambda & 1 \\ 0 & 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}.$$

The configurations δ_r^f (as well as their companions, the *mixed bar codes* not discussed in this paper) and the collections $\mathcal{J}_r(f), r \geq 0$ are *computer friendly* in the sense that for a simplicial complex and a simplicial map can be calculated by computer implementable algorithms.

On one side these invariants refine basic algebraic topology invariants of X and $(X; \xi_f)$ (Betti numbers or Novikov-Betti numbers, homology or Novikov homology, monodromy). On the other side, they are closed to the dynamical elements (rest points, instantons, closed trajectories) of a flow on X which has f as a Lyapunov map and permit to detect the presence and get information about the cardinality of such elements.

The configuration δ_r^f is a configuration of points in the complex plane, each such point corresponding to a pair of critical values of f (*i.e.* bar codes in the terminology of [2]) whose multiplicity has homological interpretation. The configuration $\hat{\delta}_r^f$ is a configuration of vector spaces or modules indexed by complex numbers with the vector space or module $\hat{\delta}_r^f(z)$ of dimension or

rank equal to $\delta_r^f(z)$ and specifying a piece of the homology $H_r(X)$ or Novikov homology $H_r^N(X, \xi_f)$. The Jordan cells $\mathcal{J}_r(f)$ are pairs (λ, k) each providing a Jordan matrix which appears in the Jordan decomposition of the r-monodromy of ξ_f.

In contrast with the classical Morse-Novikov theory concerned with critical points of f, instantons and periodic orbits of $grad_g f$ for X a smooth manifold and f a Morse real or angle valued map, the configurations δ_r^f, $\hat{\delta}_r^f$ and the Jordan cells $\mathcal{J}_r$, associated to f in AMN-theory,

– are defined for spaces X and maps f considerably more general than manifolds and Morse maps,

– are computable by effective algorithms when X is a finite simplicial complex and f simplicial map,

– enjoy robustness to C^0-perturbation and satisfy Poincaré duality.

This paper summarizes the definitions and the properties of the invariants δ_r^f, $\hat{\delta}_r^f$, $\hat{\hat{\delta}}_r^f$, $\mathcal{J}_r(f)$ in AMN-theory and addresses only the first aspect of the theory, the algebraic topology aspect. It also indicates a few mathematical applications (Section 6). The results are stated in Section 4. Details for the proofs are contained in [4–6] and partially in [7] where the computational aspects of these invariants are also addressed.

A book in preparation with the same title cf. [9] will present these invariants and their additional companions *the mixed bar codes* and will explain their relation with Morse-Novikov theory and with *Persistence Theory* = a collection of topological concepts of recent interest in Data Analysis.

2. PRELIMINARY DEFINITIONS

2.1. Configurations

Let X be a topological space and κ a fixed field. A configuration of points in X is a map $\delta : X \to \mathbb{Z}_{\geq 0}$ with finite support and a configuration of κ-vector spaces or of free $\kappa[t^{-1}, t]$-modules indexed by the points in X is a map $\hat{\delta}$ defined on X with values κ-vector spaces or free $\kappa[t^{-1}, t]$-modules with finite support. A point $x \in X$ is in the support of δ if $\delta(x) \neq 0$ and in the support of $\hat{\delta}$ if $\hat{\delta}(x)$ is of dimension or of rank different from 0. The non-negative integer $\sum_{x \in X} \delta(x)$ is referred to as the *cardinality* of δ. One denotes by $\mathcal{C}_N(X)$ the set of configurations of cardinality N.

One says that the configuration $\hat{\delta}$ refines the configuration δ if $\dim \hat{\delta}(x)$ = $\delta(x)$.

If $\kappa = \mathbb{C}$ one can consider also configurations with values in Hilbert modules of finite type over a von Neumann algebra, in our discussion always $\mathbb{L}^\infty(\mathbb{S}^1)$,

the finite von Neumann algebra obtained by the von Neumann completion of the group ring $\mathbb{C}[\mathbb{Z}]$ which is exactly $\mathbb{C}[t^{-1}, t]$.

Let V be a finite dimensional vector space over κ a field or a free f.g. $\kappa[t^{-1}, t]$-module or a finite type Hilbert module over $L^{\infty}(\mathbb{S}^1)$. Consider the set $\mathcal{P}(V)$ of subspaces of V, split free submodules of V, closed Hilbert submodules of V respectively.

One denotes by $\mathcal{C}_V(X)$ the set of configurations with values in $\mathcal{P}(V)$ which satisfy the property that the induced map $I_{\hat{\delta}} : \oplus_{x \in X} \hat{\delta}(x) \to V$ is an isomorphism. An element of $\mathcal{C}_V(X)$ will be denoted by $\hat{\hat{\delta}}$ rather than $\hat{\delta}$ to emphasize the additional properties.

The sets $\mathcal{C}_N(X)$ and $\mathcal{C}_V(X)$ carry natural topologies, referred to as the *collision topology*. One way to describe these topologies is to specify for each δ or $\hat{\delta}$ a system of *fundamental neighborhoods*.

If δ has as support the set of points $\{x_1, x_2, \cdots x_k\}$, a fundamental neighborhood $\mathcal{U}$ of δ is specified by a collection of k disjoint open neighborhoods $U_1, U_2, \cdots, U_k$ of $x_1, \cdots x_k$, and consists of $\{\delta' \in \mathcal{C}_N(X) \mid \sum_{x \in U_i} \delta'(x) = \delta(x_i)\}$. Similarly if $\hat{\hat{\delta}}$ has as support the set of points $\{x_1, x_2, \cdots x_k\}$ with $\hat{\hat{\delta}}(x_i) = V_i \subseteq V\}$, a fundamental neighborhood $\mathcal{U}$ of $\hat{\hat{\delta}}$ is specified by a collection of k disjoint open neighborhoods $U_1, U_2, \cdots U_k$ of $x_1, \cdots x_k$, and consists of configuration $\hat{\hat{\delta}}'$ which satisfy the following:

a) for any $x \in U_i$ one has $\hat{\hat{\delta}}'(x) \subset V_i$,

b) the map $I_{\hat{\hat{\delta}}}(\oplus_{x \in U_i} \hat{\hat{\delta}}'(x)) = V_i$.

Note that

Observation 2.1.

1. $\mathcal{C}_N(X)$ identifies to the N-fold symmetric product X^N / Σ_N [2] of X and if X is a metric space with distance D then the collision topology is the same as the topology defined by the metric $\underline{D}$ on X^N / Σ_N induced from the distance D. This induced metric is referred to as the *canonical metric* on $\mathcal{C}_N(X)$.

2. If $X = \mathbb{C}$ then $\mathcal{C}_N(X)$ identifies to the degree N-monic polynomials with complex coefficients and if $X = \mathbb{C} \setminus 0$ to the degree N-monic polynomials with non zero free coefficient. To the configuration δ whose support consists of the points $z_1, z_2, \cdots z_k$ with $\delta(z_i) = n_i$ one associates the monic polynomial $P^{\delta}(z) = \prod_i (z - z_i)^{n_i}$. Then as topological spaces $\mathcal{C}_N(\mathbb{C})$ identifies to $\mathbb{C}^N$ and $\mathcal{C}_N(\mathbb{C} \setminus 0)$ to $\mathbb{C}^{N-1} \times (\mathbb{C} \setminus 0)$.

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[2] Σ_N is the group of permutations of N elements.

3. If $X = \mathbb{T} := \mathbb{R}^2/\mathbb{Z}$ is the quotient of $\mathbb{R}^2$ by the action $\mu(n, (a,b)) = (a + 2\pi n, b + 2\pi n)$, then the space $\mathbb{T}$ can be identified to $\mathbb{C} \setminus 0$ by $\langle a, b \rangle \to e^{ia + (b-a)}$ and $\mathcal{C}_N(\mathbb{T})$ and $\mathcal{C}_N(\mathbb{C} \setminus 0)$ are homeomorphic. Here $\langle a, b \rangle$ denotes the μ-orbit of (a,b).

4. The *canonical metrics* $\underline{D}$ on $\mathcal{C}_N(\mathbb{R}^2)$ or $\mathcal{C}_N(\mathbb{T})$ refer to the metrics derived from the complete Euclidean metric D on $\mathbb{R}^2$ or $\mathbb{R}^2/\mathbb{Z}$. Both these metrics are complete.

2.2. **Tame maps**

A space X is an ANR if any closed subset A of a metrizable space B homeomorphic to X has a neighborhood U which retracts to A, cf. [16, Chapter 3]. Any space homeomorphic to a locally finite simplicial complex or to a finite dimensional topological manifold or an infinite dimensional manifold (*i.e.* a paracompact separable Hausdorff space locally homeomorphic to the infinite dimensional separable Hilbert space or to the Hilbert cube $[0,1]^{\infty\ [3]}$) is an ANR.

1. A continuous proper map $f : X \to \mathbb{R}$, X an ANR [4] is *weakly tame* if for any $t \in \mathbb{R}$, the level $f^{-1}(t)$ is an ANR. Therefore for any bounded or unbounded closed interval I the space $f^{-1}(I)$ is an ANR.

2. The number $t \in \mathbb{R}$ is a *regular value* if there exists $\epsilon > 0$ small s.t. for any $t' \in (t - \epsilon, t + \epsilon)$ the inclusion $f^{-1}(t') \subset f^{-1}(t - \epsilon, t + \epsilon)$ is a homotopy equivalence. A number t which is not regular value is a *critical value*. In different words the homotopy type of the $t-$level does not change in the neighborhood of a regular value and does change in any neighborhood of a critical value. One denotes by $Cr(f)$ the collection of critical values of f.

3. The map f is called *tame* if weakly tame and in addition:
   i) The set of critical values $Cr(f) \subset \mathbb{R}$ is discrete,
   ii) The number $\epsilon(f) := \inf\{|c - c'| \mid c, c' \in Cr(f), c \neq c'\}$ satisfies $\epsilon(f) > 0$.
   If X is compact then (i) implies (ii).

4. An ANR for which the set of tame maps is dense in the space of all maps w.r. to the fine-C^0 topology is called a *good ANR*.
   There exist compact ANR's (actually compact homological n-manifolds, cf. [13]) with no co-dimension one subsets which are ANR's, hence compact ANR's which are not *good*.

The reader should be aware of the following rather obvious facts.

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[3] product of countable many copies of the interval $[0,1]$

[4] The properness of f rules out infinite dimensional Hilbert manifolds but not Hilbert cube manifolds.

Observation 2.2.

1. If f is a weakly tame map then the compact ANR $f^{-1}([a,b])$ has the homotopy type of a finite simplicial complex (cf. [19]) and therefore has finite dimensional homology w.r. to any field κ.

2. If X is a locally finite simplicial complex and f is linear on each simplex then f is weakly tame with the set of critical values discrete. Critical values are among the values f takes on vertices. If in addition X is compact then f is tame. If M is a smooth manifold and f is proper smooth map with all critical points of finite codimension, in particular f is a Morse map, then f is weakly tame and when M is compact f is tame.

3. If X is homeomorphic to a compact simplicial complex or to a compact topological manifold then the set of tame maps is dense in the set of all continuous maps equipped with the C^0-topology ($=$ compact open topology). The same remains true if X is a compact Hilbert cube manifold defined in the next section. In particular, all these spaces are good ANR's.

4. On a smooth manifold the Morse functions are dense in the space of all continuous function w.r. to the fine C^0-topology and are generic in any C^r-topology, $r \geq 2$.

2.3. **Algebraic topology**

Let κ be a field. For an ANR X denote by $H_r(X)$ the (singular) homology with coefficients in κ; this is a κ-vector space which when X is compact, it is finite dimensional by [19].

Denote by $\beta_r(X) := \beta_r(X;\kappa) = \dim H_r(X)$ $r \geq 0$ referred below as the r-th Betti number and by $\chi(X) = \chi(X;\kappa) = \sum_r (-1)^r \beta_r(X)$ the Euler characteristic with coefficients in κ.

For a pair $(X, \xi \in H^1(X;\mathbb{Z}))$, X a compact ANR and ξ a degree one integral cohomology class, consider $\pi : \tilde{X} \to X$ an infinite cyclic cover associated to ξ (unique up to isomorphism), and let $\tau : \tilde{X} \to \tilde{X}$ be the generator of the group of deck transformations (the infinite cyclic group $\mathbb{Z}$).

The space $\tilde{X}$ is a locally compact ANR and the κ-vector space $H_r(\tilde{X})$ is a finitely generated $\kappa[t^{-1}, t]$-module with the multiplication by t given by the isomorphism $T_r : H(\tilde{X}) \to H_r(\tilde{X})$ induced by the homeomorphism τ. The submodule of torsion elements of $H_r(\tilde{X})$, denoted by $V_r(X;\xi)$, when regarded as a κ-vector space is finite dimensional and the $\kappa[t^{-1}, t]$-module $H_r(\tilde{X})/V_r(X;\xi)$ is free of finite rank.

The isomorphism class of the $\kappa[t^{-1}, t]$-module $V_r(X;\xi)$, equivalently of the pair $(V_r(X;\xi), T_r)$ with $V_r(X;\xi)$ viewed as a κ-vector space with a linear auto-

morphism T_r, is referred to as the r-th monodromy. The free $\kappa[t^{-1}, t]$-module $H_r^N(X : \xi) := H_r(\tilde{X})/V_r(X; \xi)$ is referred below as the Novikov homology in dimension r, and its rank as the r-Novikov–Betti number and denoted by $\beta_r^N(X; \xi)$.

If $\kappa = \mathbb{C}$ is the field of complex numbers then the ring $\mathbb{C}[t^{-1}, t]$, equivalently the group algebra $\mathbb{C}[\mathbb{Z}]$, has a canonical completion to the finite von-Neumann algebra $L^\infty(\mathbb{S}^1)$ and the module $H_r^N(X; \xi)$ to a finite type $L^\infty(\mathbb{S}^1)$-Hilbert module, of von-Neumann dimension $\beta_r^B(X; \xi)$. The completion of $H_r^N(X; \xi)$ is exactly the L_2-homology $H_r^{L_2}(\tilde{X})$, cf. [5]. The completion of $H_r^N(X; \xi)$ is referred to as the von Neumann completion, and depends a priory on additional data such as: a Riemannian metric when X a compact smooth manifold, a triangulation when X is a finite simplicial complex or more algebraically, an inner $\mathbb{C}[t^{-1}, t]$-product on $H_r^N(X; \xi)$, but all types of such different data lead to isomorphic $L^\infty(\mathbb{S}^1)$-Hilbert modules, cf. [5].

3. THE CONFIGURATIONS δ_r^f, $\hat{\delta}_r^f$, $\hat{\hat{\delta}}_r^f$ AND THE SET OF JORDAN CELLS $\mathcal{J}_r(f)$

Let $f : X \to \mathbb{R}$ be a a proper continuous map, X an ANR and κ be a fixed field. Denote by:

– X_a, the sub level $X_a := f^{-1}(-\infty, a])$,
– X^b, the super level $X^b := f^{-1}([b, \infty))$,
– $\mathbb{I}_a^f(r) := img(H_r(X_a) \to H_r(X)) \subseteq H_r(X)$,
– $\mathbb{I}_f^b(r) := img(H_r(X^b) \to H_r(X)) \subseteq H_r(X)$,
– $\mathbb{F}_r^f(a, b) := \mathbb{I}_a^f(r) \cap \mathbb{I}_f^b(r) \subseteq H_r(X)$, $F_r^f(a, b) = \dim \mathbb{F}_r^f(a, b)$.

Observation 3.1.
1. If $a' \leq a$ and $b \leq b'$ then $\mathbb{F}_r^f(a', b') \subseteq \mathbb{F}_r^f(a, b)$.
2. If $a' \leq a$ and $b \leq b'$ then $\mathbb{F}_r^f(a', b) \cap \mathbb{F}_r^f(a, b') = \mathbb{F}_r^f(a', b')$.
3. $\dim F_r(a, b) < \infty$ (cf. [4] Proposition 3.4).
4. $\sup_{x \in X} |f(x) - g(x)| < \epsilon$ implies $\mathbb{F}^g(a - \epsilon, b + \epsilon) \subseteq \mathbb{F}_r^f(a, b)$.
5. If f is weakly tame and number $a \in \mathbb{R}$ is a *regular value* then there exists $\epsilon > 0$ so that for any $0 \leq t, t' < \epsilon$ the inclusions $\mathbb{I}_{(a-t)}^f(r) \subseteq \mathbb{I}_{(a+t')}^f(r)$ and $\mathbb{I}_f^{(a-t')}(r) \supseteq \mathbb{I}_f^{(a+t)}(r)$ are isomorphisms for all r.

A set $B \subset \mathbb{R}^2$ of the form $B = (a', a] \times [b, b')$ with $a' < a, b < b'$ is called *box*, cf. Fig. 1. For $(a, b) \in \mathbb{R}^2$ and $\epsilon > 0$ denote by $B(a, b; \epsilon)$ the box $B(a, b; \epsilon) := (a - \epsilon, a] \times [b, b + \epsilon)$. To the box B we assign the vector space

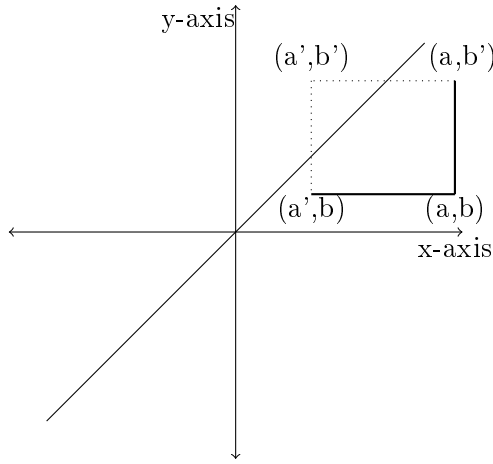$$\mathbb{F}_r^f(B) := \mathbb{F}_r^f(a, b)/\mathbb{F}_r^f(a', b) + \mathbb{F}_r^f(a, b')$$

Fig. 1. The box $B := (a', a] \times [b, b') \subset \mathbb{R}^2$.

of dimension
$$F_r^f(B) := \dim \mathbb{F}_r^f(B).$$
In view of Observation 3.1 item 3 $F_r^f(B) < \infty$ and in view of Observation 3.1 item 2
$$F_r^f(B) := F_r^f(a, b) + F^f(a', b') - F_r^f(a', b) - F^f(a, b').$$

For $a'' < a' < a, b < b' < b''$ and $B'' := (a'', a] \times [b, b'')$, $B' := (a', a] \times [b, b)$, the inclusion of vector spaces $(\mathbb{F}_r^f(a'', b) + \mathbb{F}_r^f(a, b'')) \subseteq (\mathbb{F}_r^f(a', b) + \mathbb{F}_r^f(a, b'))$ induces the canonical surjective linear map $\pi_{B'', r}^{B'} : \mathbb{F}_r^f(B'') \to \mathbb{F}_r^f(B')$.

For $0 < \epsilon' < \epsilon$ consider $B(a, b; \epsilon') \subset (a - \epsilon, a] \times [b, b + \epsilon') = B_1 \subset B(a, b; \epsilon)$ and $B(a, b; \epsilon') \subset (a - \epsilon', a] \times [b, b + \epsilon] = B_2 \subset B(a, b; \epsilon)$. One has
$$\pi_{\epsilon, r}^{\epsilon'} = \pi_{B_1, r}^{B(a, b; \epsilon')} \cdot \pi_{B(a, b; \epsilon), r}^{B_1} = \pi_{B_2, r}^{B(a, b; \epsilon')} \cdot \pi_{B(a, b; \epsilon), r}^{B_2}.$$

Consider the diagram

$$\mathbb{F}_r^f(a, b)$$

$$\pi_{(a,b),r}^{\epsilon} \swarrow \qquad \qquad \searrow \pi_{(a,b),r}^{\epsilon'}$$

$$\mathbb{F}_r^f(B(a, b; \epsilon)) \xrightarrow{\quad \pi_\epsilon^{\epsilon'} \quad} \mathbb{F}_r^f(B(a, b; \epsilon'))$$

and denote by $\hat{\delta}_r^f(a, b)$ and $\pi_r(a, b)$ the vector space

$$\boxed{\hat{\delta}_r(a, b) = \varinjlim_{\epsilon \to 0} \mathbb{F}_r^f(B(a, b; \epsilon))}$$

and the surjective linear map

$$\boxed{\pi_r(a, b) : \mathbb{F}_r^f(a, b) \to \hat{\delta}_r(a, b) = \varinjlim_{\epsilon \to 0} \pi_{(a,b)}^\epsilon.}$$

The space $\hat{\delta}_r(a, b)$ is of finite dimension since $\mathbb{F}_r^f(a, b)$ is and denote this dimension by

$$\boxed{\delta_r^f(a, b) = \dim \hat{\delta}_r^f(a, b)}.$$

In view of Observation 3.1 item 4 one proposes the following definition.

Definition 3.2. A real number t is a **homologically regular value** (w.r. to the field κ) if there exists $\epsilon(t) > 0$ s.t. for any $0 < \epsilon < \epsilon(t)$ the inclusions $\mathbb{I}_{t-\epsilon}(r) \subseteq \mathbb{I}_t(r) \subseteq \mathbb{I}_{t+\epsilon}(r)$ and $\mathbb{I}^{t-\epsilon}(r) \supseteq \mathbb{I}^t(r) \supseteq \mathbb{I}^{t+\epsilon}(r)$ are equalities and **homologically critical value** if not a **homological regular value** and let $CR(f)$ be the set of homological critical values.

By Observation 3.1 item 5, f weakly tame implies $CR(f) \subseteq Cr(f)$.

Observation 3.3 (cf. [4]).
If X is an ANR and f is a continuous proper map then $CR(f)$ is a discrete set.
If $\delta_r^f(a, b) \neq 0$ then $a, b \in CR(f)$.
If f is tame and $\delta_r^f(a, b) \neq 0$ then $\hat{\delta}_r^f(a, b) = \mathbb{F}_r^f(B(a, b; \epsilon))$ for any $\epsilon < \epsilon(f)$.[5]

The configurations in the case of a real valued map

Suppose $f : X \to \mathbb{R}$ with X compact ANR and f continuous. The assignment $\mathbb{R}^2 \ni (a, b) \rightsquigarrow \delta_r^f(a, b)$ defined above is a *configuration of points* in $\mathbb{R}^2 \equiv \mathbb{C}$, which determines and is determined by a monic polynomial $P_r^f(z)$ whose roots are the points in the support of δ_r^f with multiplicities the values of δ_r^f, and the assignment $\hat{\delta}_r^f$ is a *configuration of vector spaces* which refines δ_r^f.

If $\kappa = \mathbb{R}$ or $\mathbb{C}$ and $H_r(X)$ is equipped with a scalar product then the canonical splitting $s_r(a, b) : \hat{\delta}_r^f(a, b) \to \mathbb{F}_r(a, b)$ of $\pi_r(a, b) : \mathbb{F}_r(a, b) \to \hat{\delta}_r^f(a, b)$ given by the orthogonal complement of $\ker \pi_r(a, b)$ realizes $\hat{\delta}_r^f(a, b)$ as a subspace

$$\boxed{\hat{\hat{\delta}}_r^f(a, b) := s_r(a, b)(\hat{\delta}_r^f(a, b)) \subseteq \mathbb{F}_r(a, b) \subseteq H_r(X)}.$$

It turns out that the points $(a, b) \in \mathrm{supp}\ \delta_r^f$ with $a \leq b$ are exactly the closed r-bar codes $[a, b]$ and with $a > b$ are exactly the $(r-1)-$open bar codes (b, a) defined in [10] and [2] for the level persistence of f.

Note: One can view the configurations δ_r^f and $\hat{\delta}_r^f$ in analogy with the configuration δ^T of eigenvalues with multiplicity, and the configuration $\hat{\delta}^T$ of corresponding generalized eigenspaces, associated to a linear map $T : V \to$

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[5] This observation holds also for f continuous with the appropriate definition of $\epsilon(f)$.

V, V a finite dimensional complex vector space. The comparison provides remarkable similarities which deserve to be inspected in case of a compact smooth Riemannian manifold and of a Morse function.

The configurations in the case of an angle valued map

Suppose $f : X \to \mathbb{S}^1$ with X compact. Let $\tilde{f} : \tilde{X} \to \mathbb{R}$ be an infinite cyclic cover of f, and consider the homeomorphism $\tau : \tilde{X} \to \tilde{X}$ provided by the positive generator of the group of deck transformation $\mathbb{Z}$; hence $\tilde{f} \cdot \tau = \tilde{f} + 2\pi$. The map τ induces the isomorphism $T_r : H_r(\tilde{X}) \to H_r(\tilde{X})$ which restricts to $T_r : \mathbb{F}_r^{\tilde{f}}(a, b) \to \mathbb{F}_r^{\tilde{f}}(a + 2\pi, b + 2\pi)$ and induces the isomorphism $T_r : \hat{\delta}_r^f(a, b) \to \hat{\delta}_r^f(a + 2\pi, b + 2\pi)$.

Consider the quotient space $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}$ identified to $\mathcal{C} \setminus 0$ by $\langle a, b \rangle \to e^{ia+(b-a)}$, cf. Subsection 2.1 and define

$$\boxed{\delta_r^f(\langle a, b \rangle) = \delta_r^f(z) := \delta_r^{\tilde{f}}(a, b)},$$

$$\boxed{\hat{\delta}_r^f(\langle a, b \rangle) = \hat{\delta}_r^f(z) := \oplus_{n \in \mathbb{Z}} \hat{\delta}_r^{\tilde{f}}(a + 2n\pi, b + 2n\pi)}$$

and

$$T_r(\langle a, b \rangle) = \oplus_{n \in \mathbb{Z}} T_r(a + 2n\pi, b + 2n\pi) : \hat{\delta}_r^f(\langle a, b \rangle) \to \hat{\delta}_r^f(\langle a, b \rangle).$$

The pair $(\hat{\delta}_r^f(\langle a, b \rangle), T_r(\langle a, b \rangle))$ defines a $\kappa[t^{-1}, t]$-module which is free

It turns out that the points $z = e^{ia+(b-a)} \in supp\delta_r^f$ with $a \leq b, a \in [0, 2\pi)$ are exactly the closed r-bar codes $[a, b]$ and with $a > b, a \in [0, 2\pi)$ are exactly the the $(r-1)$-open bar codes (b, a) defined in [2].

As already pointed out in Subsection 2.3, when $\kappa = \mathbb{C}$ the algebra $\mathbb{C}[t^{-1}, t]$ can be canonically completed to the finite von-Neumann algebra $L^\infty(\mathbb{S}^1)$. Additional data (for example a $\mathbb{C}[t^{-1}, t]$-inner product on $H_r^N(X; \xi)$, or a Riemannian metric on X when X is a Riemannian manifold, or a triangulation of X when X is a simplicial complex) lead to a completion of $H_r^N(X; \xi)$ as Hilbert $L^\infty(\mathbb{S}^1)$-module, the L_2-homology $H^{L_2}(\tilde{X})$, and of $\hat{\delta}_r^f(\langle a, b \rangle)$ as a closed Hilbert submodule of $H^{L_2}(\tilde{X})$. The procedure of such completions is described in [5, Section 2] and called the *von-Neumann completion*.

The assignments δ_r^f, $\hat{\delta}_r^f$, and $\hat{\delta}_r^f$ are configurations of points with multiplicities, free $\kappa[t^{-1}, t]$-modules and $L^\infty(\mathbb{S}^1)$-Hilbert modules respectively.

The Jordan cells for an angle valued map

For $f : X \to \mathbb{S}^1$ tame and $\theta \in \mathbb{S}^1$ denote by $X_\theta := f^{-1}(\theta)$ and by $\overline{X}_\theta$ the two sided compactification of $f^{-1}(\mathbb{S}^1 \setminus \theta)$ by $f^{-1}(\theta)$, called in [6] the *cut of f at θ*. The space $\overline{X}_\theta$ is homeomorphic to the compact space $\tilde{f}^{-1}[t, t + 2\pi]$ for

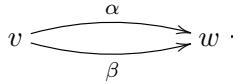any $t \in \mathbb{R}$ with $p(t) = \theta$. The inclusions

$$X_\theta = \tilde{f}^{-1}(t) \xrightarrow{\subset} \overline{X}_\theta = f^{-1}([t, t+2\pi] \xleftarrow{\supset} X_\theta = f^{-1}(t+2\pi)$$

induce in homology the linear map

$$H_r(X_\theta) \xrightarrow{a} H_r(\overline{X}_\theta) \xleftarrow{b} H_r(X_\theta)$$

which can be regarded as a *linear relation*, cf. [6,7], or as a graph representation of the oriented graph G_2.

The oriented graph G_2 has two vertices v, w and two oriented edges from v to w denoted by α and β as indicated below

$$v \underset{\beta}{\overset{\alpha}{\rightrightarrows}} w \; .$$

A linear representation ρ of G_2 is provided by two vector spaces of finite dimension V and W associated to v and w and two linear maps $a, b : V \to W$ associated to the edges α, β. The concept of isomorphism of representations direct sum of representations and indecomposable representations are obvious and, as in the case of an arbitrary finite oriented graph, each representation has a decomposition as sum of a unique collection (up to permutation) of indecomposables; the decomposition is not unique. If κ is algebraically closed the list of *indecomposables* can be recovered from an old theorem of Kronecker (a proof of Kronecker theorem can be found in [1]) and is provided below.

1. Representation denoted by $\rho^+(r)$ has $V = \kappa^r, W = \kappa^{r+1}$, $a = \begin{bmatrix} Id_r \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ Id_r \end{bmatrix}$.

2. Representation denoted by $\rho^-(r)$ has $V = \kappa^{r+1}, W = \kappa^r$, $a = \begin{bmatrix} Id_r & 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 & Id_r \end{bmatrix}$.

3. Representation denoted by (λ, k) called Jordan cells, has $V = \kappa^r, W = \kappa^r$, $a = T(\lambda, k)$, $b = \begin{bmatrix} Id_k \\ 0 \end{bmatrix}$.

One defines the set $\mathcal{J}_r(f, \theta)$ the collection of the Jordan cells associated to the G_2 representation given by

$$H_r(X_\theta) \xrightarrow{a} H_r(\overline{X}_\theta) \xleftarrow{b} H_r(X_\theta) \; .$$

4. THE RESULTS

As noticed in Section 2 the configuration δ_r^f defined in Section 3 can be equally regarded as a monic polynomial $P_r^f(z)$ whose zeros are the complex

numbers $z \in supp\ \delta_r^f$ with multiplicities equal to $\delta_r^f(s)$.

Results about real valued maps

THEOREM 4.1 (Topological results). *Suppose X compact ANR and $f : X \to \mathbb{R}$ continuous. Then the following holds.*

1. *If $P_r^f(z) = 0$, equivalently $\delta_r^f(z) \neq 0$ with $z = (a + ib)$, then $a, b \in CR(f)$.*
2. *The configuration $\delta_r^f \in \mathcal{C}_{\dim H_r(X)}(\mathbb{C})$, the configuration δ_r^f satisfies $\oplus_{z \in \mathbb{C}} \hat{\delta} f_r(z) \simeq H_r(X)$ and if $\kappa = \mathbb{R}$ or $\mathbb{C}$ and $H_r(X)$ is equipped with a Hilbert space structure (i.e. a scalar product) then the configuration $\hat{\hat{\delta}}^f(r) \in \mathcal{C}_{H_r(X)}(\mathbb{C})$ and satisfies $\hat{\hat{\delta}}_r^f(z) \perp \hat{\hat{\delta}}_r^f(z)$ for $z \neq z'$.*
3. *For f in an open and dense subset of the space of continuous real valued maps equipped with the compact open topology one has $\delta_r^f(z) = 0$ or 1.*

THEOREM 4.2 (Stability). *Suppose X is a compact ANR.*

1. *The assignment $f \rightsquigarrow \delta_r^f$ provides a continuous map from the space of real valued maps equipped with the compact open topology to the space of configurations $\mathcal{C}_{b_r}(\mathbb{R}^2) = \mathcal{C}_{b_r}(\mathbb{C}) \simeq \mathbb{C}^{b_r}$, $b_r = \dim H_r(X)$, equipped with the collision topology, equivalently to the space of monic polynomials of degree b_r.*

Moreover, with respect to the canonical metric $\underline{D}$ (cf. Observation 2.1) on the space of configurations $\mathcal{C}_{b_r}(\mathbb{R}^2)$, and the metric $D(f, g) := ||f - g||_\infty = sup_{x \in X}|f(x) - g(x)|$ on the space of continuous maps one has

$$\underline{D}(\delta^f, \delta^g) < 2D(f, g).$$

2. *If $\kappa = \mathbb{R}$ or C, and $H_r(X)$'s are equipped with scalar products then the assignment $f \rightsquigarrow \hat{\hat{\delta}}_r^f$ is also continuous provided that $\mathcal{C}_{H_r(X)}(\mathbb{C})$ is equipped with the collision topology described in Subsection 2.1.*

THEOREM 4.3 (Poincaré Duality). *Suppose X is a closed topological manifold of dimension n which is κ-orientable [6] and $f : X \to \mathbb{R}$ is a continuous map. Then:*

1. $\delta_r^f(a, b) = \delta_{n-r}^f(b, a)$.

2. *If $\kappa = \mathbb{R}, \mathbb{C}$ and the vector spaces $H_r(X)$'s are equipped with scalar products then the canonical isomorphism induced by the Poincaré duality and the scalar products, $PD_r : H_r(X) \to H_{n-r}(X)$, intertwines the configuration $\hat{\hat{\delta}}_r^f$ and $\hat{\hat{\delta}}_{n-r}^f \cdot \tau$ where $\tau(a, b) = (b, a)$. In particular if X is a closed Riemannian manifold, hence $H_r(X)$ identifies to the space of harmonic $(n - r)-$ differential forms, then the Hodge star operator intertwines $\hat{\hat{\delta}}_r^f$ with $\hat{\hat{\delta}}_{n-r}^f \cdot \tau$.*

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[6] when κ is of characteristic 2 all manifolds are κ-orientable if not of characteristic 2 only the orientable manifolds are κ-orientable.

Results about angle valued maps

Let $f : X \to \mathbb{S}^1$ be a continuous map, X compact ANR, and let $\xi := \xi_f \in H^1(X; \mathbb{Z})$ be the integral cohomology class represented by f. Let $\tilde{X} \to X$ be an infinite cyclic cover associated to ξ. If $\kappa = \mathbb{C}$ let $H_r^{L^2}(\tilde{X})$ be the von–Neumann completion of $H^N(X; \xi)$ as described in [5].

THEOREM 4.4 (Topological results). *Suppose X compact ANR and $f : X \to \mathbb{S}^1$ continuous map. Then the following holds.*

1. *If $P_r^f(z) = 0$, equivalently $\delta_r^f(z) \neq 0$ with $z = e^{ia+(b-a)}$, then e^{ia}, e^{ib} are in $CR(f)$.*

2. *The configuration $\delta_r^f(z) \in \mathcal{C}_{\beta_r^N(X;\xi_f)}(\mathbb{C} \setminus 0)$, the configuration $\hat{\delta}_r^f$ satisfies $\oplus \hat{\delta}_r^f \simeq H_r^N(X; \xi)$ and if $\kappa = \mathbb{C}$ then the configuration $\hat{\hat{\delta}}_r^f \in \mathcal{C}_{H^{L^2}(\tilde{X})}(\mathbb{C} \setminus 0)$ and satisfies $\hat{\hat{\delta}}_r^f(z) \perp \hat{\hat{\delta}}_r^f(z')$ for $z \neq z'$.*

3. *If $C_\xi(X, \mathbb{S}^1)$ denotes the set of continuous maps in the homotopy class determined by ξ equipped with the compact open topology then for f in an open and dense set of maps of $C_\xi(X, \mathbb{S}^1)$ one has $\delta^f(z) = 0$ or 1.*

THEOREM 4.5 (Stability). *Suppose X is a compact ANR and $\xi \in H^1(X; \mathbb{Z})$. Then:*

1. *The assignment*
$$C(X, \mathbb{S}^1)_\xi \ni f \rightsquigarrow \delta_r^f \in \mathcal{C}_{\beta_r^N(X;\xi)}(\mathbb{C} \setminus 0) \equiv \mathcal{C}_{\beta_r^N(X;\xi)}(\mathbb{R}^2)$$
equivalently
$$C(X, \mathbb{S}^1)_\xi \ni f \rightsquigarrow P_r^f(z) \in \mathbb{C}^{\beta_r^N(X;\xi)} \times (\mathbb{C} \setminus 0)$$
provides a continuous map from $C_\xi(X, \mathbb{S}^1)$, the space of continuous maps in the homotopy class determined by ξ equipped with the compact open topology, to the space of configurations $\mathcal{C}_{\beta_r^N(X;\xi)}(\mathbb{C} \setminus 0)$ equivalently $\mathbb{C}^{\beta_r^N(X;\xi)} \times (\mathbb{C} \setminus 0)$.

Moreover, with respect to the canonical metric $\underline{D}$ on $C_{\beta_r^N(X;\xi)}(\mathbb{T})$ and the complete metric $D(f, g) := \sup_{x \in X} d(f(x), g(x))$ on $C_\xi(X, \mathbb{S}^1)$, d the distance in $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, one has

$$\underline{D}(\delta^f, \delta^g) < 2\pi D(f, g).$$

2. *If $\kappa = \mathbb{C}$ and the space of configurations $\mathcal{C}_{H_r^{L^2}(\tilde{X})}(\mathbb{C} \setminus 0)$ is equipped with the collision topology then the assignment $f \rightsquigarrow \hat{\hat{\delta}}_r^f$ is continuous.*

THEOREM 4.6 (Poincaré Duality). *Suppose M is a closed topological manifold of dimension n which is κ-orientable, $f : M \to \mathbb{S}^1$ is a continuous map and $\tilde{M}$ the infinite cyclic cover defined by ξ_f. Then one has*

1. *$\delta_r^f(\langle a, b \rangle) = \delta_{n-r}^f(\langle b, a \rangle)$, equivalently $\delta_r^f(z) = \delta_{n-r}^f(\tau z)$ with $\tau(z) = z^{-1} e^{i \ln |z|}$. Here $\langle a, b \rangle$ denotes the element of $\mathbb{T}$ represented by $(a, b) \in \mathbb{R}^2$.*

2. *If $\kappa = \mathbb{C}$ and M^n is a closed Riemannian manifold then the canonical isomorphism of $H_r^{L_2}(\tilde{M})$ to $H_{n-r}^{L_2}(\tilde{M})$ induced by the Riemannian metric (via L_2 harmonic forms and Hodge star operator) intertwines the configuration $\hat{\delta}_r^f$ and $\hat{\delta}_{n-r}^f \cdot \tau$ when regarded as configurations on $\mathbb{R}^2/\mathbb{Z} = \mathbb{T}$.*

In Section 3, for a weakly tame map $f : X \to \mathbb{S}^1$ and an angle $\theta \in \mathbb{S}^1$ we have defined the collection of Jordan cells $\mathcal{J}_r(f, \theta)$. Our definition make them computable by effective computer implementable algorithms. They have the following properties.

PROPOSITION 4.7.
1. *If $f : X \to \mathbb{S}^1$ is a weakly tame map then the set $\mathcal{J}_r(f, \theta)$ is independent on θ, so the notation $\mathcal{J}_r(f; \theta)$ can be abbreviated to $\mathcal{J}_r(f)$.*
2. *If $f_1 : X_1 \to \mathbb{S}^1$ and $f_2 : X_2 \to \mathbb{S}^1$ are two weakly tame maps and $\omega : X_1 \to X_2$ a homeomorphism s.t. $f_2 \cdot \omega$ and f_1 are homotopic then $\mathcal{J}_r(f_1) = \mathcal{J}_r(f_2)$.*

This permits to define for any pair (X, ξ) with X a space homotopy equivalent to a compact ANR and $\xi \in H^1(X; \mathbb{Z})$ the invariant $\mathcal{J}_r(X, \xi)$ by $\mathcal{J}_r(X, \xi) := \mathcal{J}_r(f)$ where $f : Y \to \mathbb{S}^1$ is a simplicial map defined on the simplicial complex Y homotopy equivalent to X by a homotopy equivalence $\omega : X \to Y$ s.t. $f \cdot \omega$ represents ξ. In view of the discussion on the topology of compact Hilbert cube manifolds such pairs (Y, ω) exist. The invariant $\mathcal{J}_r(X; \xi)$ satisfies the following.

THEOREM 4.8.
1. *If $\omega : X_1 \to X_2$ is a homotopy equivalence s.t. $\omega^*(\xi_2) = \xi_1$, $\xi_1 \in H^1(X_1, \mathbb{Z})$, $\xi_2 \in H^1(X_2, \mathbb{Z})$ and X_1 and X_2 have the homotopy type of a compact ANR then $\mathcal{J}(X_1, \xi_1) = \mathcal{J}_r(X_2, \xi_2)$.*
2. *If X is a compact ANR then $\mathcal{J}_r(X, \xi)$ are exactly the Jordan cells of the monodromy*

$$T_r(X, \xi) : V_r(X, \xi) \to V_r(X, \xi).$$

Introduce the set

$$\mathcal{J}_r(X; \xi)(u) := \{(\lambda, k) \in \mathcal{J}_r(X : \xi) \mid \lambda = u\}$$

and for a finite set S denote by $\sharp S$ the cardinality of S.

For any field κ one has the following relation between Betti numbers, Novikov Betti numbers and Jordan cells.

THEOREM 4.9. $\beta_r(X) = \beta_r^N(X, \xi) + \sharp \mathcal{J}_r(X, \xi)(1) + \sharp \mathcal{J}_{r-1}(X, \xi)(1).$

5. ABOUT THE PROOF

The proof of Theorems 4.1, 4.2 , 4.3 is contained partially in [7] and as stated in [4], of Theorems 4.4, 4.5 , 4.6 partially in [7] and as stated in [5], and of Proposition 4.7 and Theorem 4.8 in [7] and [6].

The proofs are done first for nice spaces (homeomorphic to simplicial complexes) and tame maps and then extended to an arbitrary compact ANR and arbitrary continuous map based on results on compact Hilbert cube manifolds as summarized in Theorem 5.4 below.

As far as the first step is concerned, the following propositions of various level of complexity are essential intermediate results whose proofs are contained in [4].

PROPOSITION 5.1. *Let $a' < a < a''$, $b < b''$ and B_1, B_2, and B the boxes $B_1 = (a', a] \times [b, b'')$, $B_2 = (a, a''] \times [b, b'')$ and $B = (a', a''] \times [b, b'')$ (see Fig. 2).*

1. The inclusions $B_1 \subset B$ and $B_2 \subset B$ induce the linear maps $i^B_{B_1, r}$: $\mathbb{F}_r(B_1) \to \mathbb{F}_r(B)$ and $\pi^{B_2}_{B, r} : \mathbb{F}_r(B) \to \mathbb{F}_r(B_2)$ such that the following sequence is exact

$$0 \longrightarrow \mathbb{F}_r(B_1) \xrightarrow{\ i^B_{B_1, r}\ } \mathbb{F}_r(B) \xrightarrow{\ \pi^{B_2}_{B, r}\ } \mathbb{F}_r(B_2) \longrightarrow 0 \ .$$

2. If $\kappa = \mathbb{R}$ or $\mathbb{C}$ and $H_r(X)$ is equipped with a scalar product then $\mathbb{F}_r(B)'s$ can be canonically realized as subspaces $\mathbf{H}_r(B) \subseteq H_r(M)$ with the property

$$\mathbf{H}_r(B_1) \perp \mathbf{H}_r(B_2)$$

and

$$\mathbf{H}_r(B) = \mathbf{H}_r(B_1) + \mathbf{H}_r(B_2).$$

PROPOSITION 5.2. *Let $a' < a$, $b' < b < b''$ and B_1, B_2, and B the boxes $B_1 = (a', a] \times [b, b'')$, $B_2 = (a, a''] \times [b', b)$ and $B = (a', a] \times [b', b'')$ (see Fig. 3).*

1. The inclusions $B_1 \subset B$ and $B_2 \subset B$ induce the linear maps $i^B_{B_1, r}$: $\mathbb{F}_r(B_1) \to \mathbb{F}_r(B)$ and $\pi^{B_2}_{B, r} : \mathbb{F}_r(B) \to \mathbb{F}_r(B_2)$ such that the following sequence is exact

$$0 \longrightarrow \mathbb{F}_r(B_1) \xrightarrow{\ i^B_{B_1, r}\ } \mathbb{F}_r(B) \xrightarrow{\ \pi^{B_2}_{B, r}\ } \mathbb{F}_r(B_2) \longrightarrow 0 \ .$$

2. If $\kappa = \mathbb{R}$ or $\mathbb{C}$ and $H_r(X)$ is equipped with a scalar product then

$$\mathbf{H}_r(B_1) \perp \mathbf{H}_r(B_2)$$

and

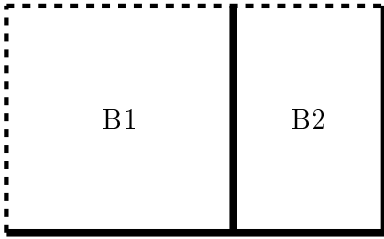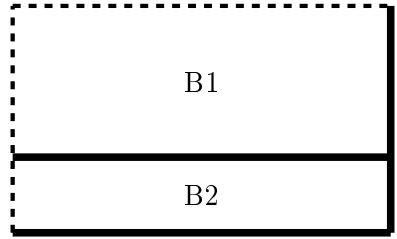$$\mathbf{H}_r(B) = \mathbf{H}_r(B_1) + \mathbf{H}_r(B_2).$$

Fig. 2

Fig. 3

PROPOSITION 5.3 (cf. Proposition 5.6 [7]). *Let $f : X \to \mathbb{R}$ be a tame map and $\epsilon < \epsilon(f)/3$. For any map $g : X \to \mathbb{R}$ which satisfies $||f - g||_\infty < \epsilon$ and $a, b \in Cr(f)$ critical values one has*

$$(1) \qquad \sum_{x \in D(a,b;2\epsilon)} \delta_r^g(x) = \delta_r^f(a,b),$$

$$(2) \qquad supp \ \delta_r^g \subset \bigcup_{(a,b) \in supp \ \delta_r^f} D(a,b;2\epsilon).$$

If $\kappa = \mathbb{R}$ or $\mathbb{C}$ and in addition $H_r(X)$ is equipped with a scalar product the above statement can be strengthen to

$$(3) \qquad x \in D(a,b;2\epsilon) \Rightarrow \hat{\hat{\delta}}_r^g(x) \subseteq \hat{\hat{\delta}}_r^f(a,b) \ \text{and} \ \oplus_{x \in D(a,b;2\epsilon)} \hat{\delta}_r^g(x) = \hat{\delta}_r^f(a,b).$$

Theorems 4.1 and 4.3 follows essentially from the first two propositions which imply that F_r is a measure on the sigma algebra generated by *boxes* with δ_r^f the *measure density*. Theorems 4.2 and 4.5, in case the source of the map f is a simplicial complex, uses essentially Proposition 5.3 and Theorems 4.3 and 4.6 use manipulation of Poincaré duality and alternative definition of $\hat{\delta}_r^f$, cf. [4]. In case of Theorem 4.6 a more elaborated manipulation involving Poincaré duality for the open manifold $\tilde{M}$, the infinite cyclic cover of $f : M \to \mathbb{S}^1$, and the description of the torsion of the $\kappa[t^{-1}, t]$ module $H_r(\tilde{M})$ are needed, cf. [5]. The proof of Theorem 4.8 involves the recognition of what in [7] and [6] is referred to as the *regular part of the linear relation* defined by the pair of linear maps

$$H_r(X_\theta) \xrightarrow{\ \ a \ \ } H_r(\overline{X}_\theta) \xleftarrow{\ \ b \ \ } H_r(X_\theta) \ .$$

Concerning the results about compact Hilbert cube manifolds used in this work recall that:

The Hilbert cube Q is the infinite product $Q = \prod_{i \in \mathbb{Z}_{\geq 0}} I_i$ with $I_i = [0, 1]$; its topology is also given by the metric $d(\overline{u}, \overline{v}) = \sum_i |u_i - v_i|/2^i$ with $\overline{u} = \{u_i \in$

$I_i, i \in \mathbb{Z}_{\geq 0}\}$ and $\overline{v} = \{v_i \in I_i, i \in \mathbb{Z}_{\geq 0}\}$. The space Q is a compact ANR and so is any $X \times Q$ for X any compact ANR.

A compact Hilbert cube manifold is a compact Hausdorff space locally homeomorphic to the Hilbert cube and is a compact ANR. The following basic results about Hilbert cube manifolds can be found in [11].

THEOREM 5.4.
1. *(R Edwards) X is a compact ANR iff $X \times Q$ is a compact Hilbert cube manifold.*
2. *(T.Chapman) Any compact Hilbert cube manifolds is homeomorphic to $K \times Q$ for some finite simplicial complex K.*
3. *(T Chapman) If $\omega : X \to Y$ is a simple homotopy equivalence between two finite simplicial complexes with Whitehead torsion $\tau(\omega) = 0$ then there exists a homeomorphism $\omega' : X \times Q \to Y \times Q$ s.t. ω' and $\omega \times id_Q$ are homotopic. If ω is only a homotopy equivalence the same conclusion holds for Q replaced by $Q \times \mathbb{S}^1$.[7]*

If we write $I^\infty = I^k \times I^{\infty-k}$ one observes that given $\epsilon > 0$ for any continuous real or angle valued map f defined on $K \times Q$, K a finite simplicial complex, there exists N large enough such that f is $\epsilon-$closed to $g \cdot \pi$ where $\pi : K \times I^\infty \to K \times I^N$ is the canonical projection on $K \times I^N$ and g is a simplicial map defined on $K \times I^N$. In particular any compact Hilbert cube manifold is a good ANR.

It can be also verified, using the definitions, that if $f : X \to \mathbb{R}$ or $\mathbb{S}^1$ is continuous, K a compact ANR and $f^K = f \cdot \pi_K$ with $\pi_K : X \times K \to X$ the canonical projection on X then $\hat{\delta}_r^{f^K}(a,b) = \oplus_{k \geq 0} \hat{\delta}_{r-k}^f(a,b) \otimes H_k(K)$ or $\hat{\delta}_r^{f^K}(\langle a,b \rangle) = \oplus_{k \geq 0} \hat{\delta}_{r-k}^f(a,b) \otimes H_k(K)$.

6. SOME APPLICATIONS

6.1. Geometric analysis

Theorem 4.1 insures that a generic continuous function provides one dimensional subspaces in homology with coefficients in a fixed field; in particular for $\kappa = \mathbb{R}$ or $\mathbb{C}$ for a closed Riemannian manifold (M, g) a generic smooth real valued function on M provides an orthonormal base (up to sign) in the space of harmonic forms.

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[7] Some partial but relevant results on the line of Theorem 5.4 were due to J. West as indicated in [11].

We expect (but have not found this result in literature) that the eigenforms of the Laplace Beltrami operators for a generic Riemannian metric in any dimension provides a similar decomposition for the smooth differential forms orthogonal to the harmonic forms. This is indeed the case in view of a result of Uhlenbeck[8] for degree zero forms and for $n = 2$ for all degree. This shows that a generic pair, Riemannian metric and smooth function, provides an orthonormal base (in Fourier sense) up to sign in the space of all differential forms; as the trigonometric functions on $\mathbb{S}^1$ provide an orthonormal base (in Fourier sense) for all smooth functions. Such base can be a useful tool in geometric analysis.

6.2. **Topology**

Observation 6.1.

1. Theorem (4.3) implies that for a closed orientable manifold of dimension n the pair of real numbers $(c, c') \in \mathrm{supp}\delta_r^f$ iff the pair $(c', c) \in \mathrm{supp}\delta_{n-r}^f$ and both pairs appear with equal multiplicity $\delta_r^f(c, c') = \delta_{n-r}^f(c', c)$.

2. Theorem 4.6 remains valid with the same proof in case M is a compact manifold with boundary $(M, \partial M)$, provided $H_r^N(\partial M; \xi_{f_{\partial M}})$ [9] vanishes for all r. In particular, under the above vanishing hypothesis, $H_r^N(M; \xi_f) \simeq H_{n-r}^N(M; \xi_f)$.

COROLLARY 6.2. *Suppose $(M^{2n}, \partial M^{2n})$ is a compact orientable manifold with boundary which has the homotopy type of a simplicial complex of dimension $\leq n$ and $\xi \in H^1(M; \mathbb{Z})$ s.t $H_r^N(\partial M; \xi_{\partial M}) = 0$ for all r. Then for any field κ :*

1. $\beta_r^N(X : \xi) = \begin{cases} 0 \text{ if } r \neq n \\ (-1)^n \chi(M_n) \text{ if } r = n \end{cases}$ *, with $\chi(M)$ the Euler -Poincaré*

characteristic with coefficients in κ.

2. $\beta_r(X) = \begin{cases} \alpha_{r-1} + \alpha_r \text{ if } r \neq n \\ \alpha_{n-1} + \alpha_n + (-1)^n \chi(M_n) \text{ if } r = n \end{cases}$ *, where α_r denotes*

the number of Jordan cells $(\lambda, k) \in J_r(M, \xi_f)$, with $\lambda = 1$.

3. *If $V^{2n-1} \subset M^{2n}$ is a compact proper sub manifold (i.e. $V \pitchfork \partial M$,[10] and $V \cap \partial M = \partial V$) representing the homology class in $H_{n-1}(M, \partial M)$ Poincaré dual to ξ_f and $H_r(V) = 0$, then the set of Jordan cells $J_r(M, \xi)$ is empty.*

Item 1 follows from Observation (6.1) and the fact that both Betti numbers and Novikov–Betti numbers calculate the same Euler–Poincaré characte-

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[8] Which claims that for a closed manifold equipped with a generic Riemannian metric the eigenvalues of the Laplace operator are simple.

[9] with $f_{\partial M}$ notation for the restriction of f to ∂M

[10] $\pitchfork$ = transversal.

ristic. Item 2 follows from Theorem 11 item c. in [7], and Item 3 from the description of Jordan cells in terms of linear relations as provided in [7] or [6].

As pointed out to us by L. Maxim, the complement $X = \mathbb{C}^n \setminus V$ of a complex hyper surface $V \subset \mathbb{C}^n, V := \{(z_1, z_2, \cdots z_n) \mid f(z_1, z_2, \cdots z_n) = 0\}$ regular at infinity, equipped with the canonical class $\xi_f \in H^1(X; \mathbb{Z})$ defined by $f : X \to \mathbb{C} \setminus 0$ is an open manifold with an integral cohomology class $\xi \in H^1(X; \mathbb{Z})$ represented by $f/|f| : X \to \mathbb{S}^1$. This manifold has compactifications to manifolds with boundary with cohomology class which satisfies the hypotheses above. Item 1 recovers a calculation of L. Maxim, cf. [17] and [15][11] that the complement of an algebraic hyper surface regular at infinity has vanishing Novikov homologies in all dimension but n.

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